C2 Exercise 1F (Mixed exercise)

Note Title 18/01/2007

1a Simplify $\dfrac{3x^4 - 21x}{3x} = x^3 - 7$

b Simplify $\dfrac{x^2 - 2x - 24}{x^2 - 7x + 6} = \dfrac{(x-6)(x+4)}{(x-6)(x-1)} = \dfrac{x+4}{x-1}$

c Simplify $\dfrac{2x^2 + 7x - 4}{2x^2 + 9x + 4} = \dfrac{(2x-1)(x+4)}{(2x+1)(x+4)} = \dfrac{2x-1}{2x+1}$

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2 Divide $3x^3 + 12x^2 + 5x + 20$ by $(x+4)$

$$
\begin{array}{rl}
 & 3x^2 + 0x + 5 \\
x+4 \,\big) & 3x^3 + 12x^2 + 5x + 20 \\
 & -(3x^3 + 12x^2) \\
\hline
 & 0x^2 + 5x \\
 & -(0x^2 + 0x) \\
\hline
 & 5x + 20 \\
 & -(5x + 20) \\
\hline
 & 0
\end{array}
$$

Check (1)

$(x+4)(3x^2+5)$

$= 3x^3 + 5x + 12x^2 + 20$

Check (2)

$f(-4) = 3(-64) + 12(16) + 5(-4) + 20$

$= 0.$

So $\dfrac{3x^3 + 12x^2 + 5x + 20}{x+4} = 3x^2 + 5$

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3 Simplify $\dfrac{2x^3+3x+5}{x+1} = 2x^2-2x+5$

$$
\begin{array}{r|l}
 & 2x^2 \quad -2x \quad +5 \\
\hline
x+1 & 2x^3 + 0x^2 + 3x + 5 \\
 & -(2x^3 + 2x^2) \\
\hline
 & \quad -2x^2 + 3x \\
 & \quad -(-2x^2 - 2x) \\
\hline
 & \qquad 5x + 5 \\
 & \qquad -(5x+5) \\
\hline
 & \qquad\qquad 0
\end{array}
$$

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4 Show that $(x-3)$ is a factor of $2x^3-2x^2-17x+15$. Hence express $2x^3-2x^2-17x+15$ in the form $(x-3)(Ax^2+Bx+C)$ where the values A, B and C are to be found.

$(x-3)$ is a factor of $f(x) \Leftrightarrow f(3)=0$.

$$\begin{aligned} f(3) &= 2(3)^3 - 2(3)^2 - 17(3) + 15 \\ &= 54 - 18 - 51 + 15 \\ &= 0 \text{ as required.} \end{aligned}$$

Now

$$\begin{array}{r} 2x^2 + 4x - 5 \\ x-3 \overline{) 2x^3 - 2x^2 - 17x + 15} \\ -(2x^3 - 6x^2) \qquad\qquad \\ \hline 4x^2 - 17x \qquad \\ -(4x^2 - 12x) \qquad \\ \hline -5x + 15 \end{array}$$

So $f(x) = (x-3)(2x^2+4x-5)$

C2 Ex1F

5 Show that $(x-2)$ is a factor of $x^3+4x^2-3x-18$. Hence express $x^3+4x^2-3x-18$ in the form $(x-2)(px+q)^2$ where the values p and q are to be found.

$$
\begin{array}{r|l}
 & x^2 + 6x + 9 \\
\hline
x-2 & x^3 + 4x^2 - 3x - 18 \\
 & -(x^3 - 2x^2) \\
\hline
 & 6x^2 - 3x \\
 & -(6x^2 - 12x) \\
\hline
 & +9x - 18
\end{array}
$$

$$x^2+6x+9 = (x+3)^2$$

So $x^3+4x^2-3x-18 = (x-2)(x+3)^2$

C2 Ex 1F

6 Factorise completely $2x^3 + 3x^2 - 18x + 8$

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 18(-2) + 8$$
$$= -16 + 12 + 36 + 8 = 40$$

$$f(2) = 2(2)^3 + 3(2)^2 - 18(2) + 8$$
$$= 16 + 12 - 36 + 8 = 0$$

$$\begin{array}{r|l}
 & 2x^2 + 7x - 4 \\
\hline
x-2 & 2x^3 + 3x^2 - 18x + 8 \\
 & -(2x^3 - 4x^2) \\
\hline
 & 7x^2 - 18x \\
 & -(7x^2 - 14x) \\
\hline
 & -4x + 8 \\
 & (-4x + 8) \\
\hline
 & 0
\end{array}$$

$$2x^2 + 7x - 4 = (2x - 1)(x + 4)$$

So $2x^3 + 3x^2 - 18x + 8$
$$= (x-2)(2x-1)(x+4)$$

C2 Ex 1F

7 Find the value of k if $(x-2)$ is a factor of $x^3-3x^2+kx-10$.

if $(x-2)$ is a factor $f(2)=0$

$f(2) = (2)^3 - 3(2)^2 + 2k - 10 = 0$

$\Rightarrow \quad 8 - 12 - 10 + 2k = 0$

$\Rightarrow \quad 2k = 14$

$\Rightarrow \quad k = 7.$

C2 Ex1F

8 Find the remainder when $16x^5 - 20x^4 + 8$ is divided by $(2x-1)$

By the remainder theorem the remainder R is given by

$$R = f(1/2) = 16\left(\tfrac{1}{2}\right)^5 - 20\left(\tfrac{1}{2}\right)^4 + 8$$

$$\Rightarrow \quad R = \frac{1}{2} - \frac{5}{4} + 8 = \frac{32 + 2 - 5}{4} = \frac{29}{4}$$

$$\Rightarrow \quad R = 7\tfrac{1}{4}$$

C2 Ex 1F

9 $f(x) = 2x^2 + px + q$. Given that $f(-3) = 0$ and $f(4) = 21$.

a find p & q.

$f(-3) = 2(-3)^2 - 3p + q = 0 \Rightarrow 18 - 3p + q = 0$ ①

$f(4) = 2(4)^2 + 4p + q = 21 \Rightarrow 32 + 4p + q = 21$ ②

Now ① − ② $\Rightarrow -14 - 7p = -21 \Rightarrow -7p = -7$

$\Rightarrow p = 1$

$p = 1 \Rightarrow 18 - 3 + q = 0 \Rightarrow q = -15$

b factorise $f(x) = 2x^2 + x - 15$

$= (2x - 5)(x + 3)$

Check $f(-3) = (-6-5)(-3+3) = (-11)(0) = 0$,

$f(4) = (8-5)(4+3) = 3 \times 7 = 21$

C2 Ex 1F

10 $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$ and $h(2) = 30$

a find the values of r & s.

$h(-1) = (-1)^3 + 4(-1)^2 + r(-1) + s = 0 \Rightarrow -1 + 4 - r + s = 0$
$\Rightarrow s = r - 3$

$h(2) = (2)^3 + 4(2)^2 + r(2) + s = 30 \Rightarrow 8 + 16 + 2r + s = 30$
$\Rightarrow s = 6 - 2r$

$\Rightarrow r - 3 = 6 - 2r \Rightarrow 3r = 9 \Rightarrow r = 3$
$\Rightarrow s = 0$

b find the remainder when $h(x)$ is divided by $(3x - 1)$.

$R = h\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) = \frac{1}{27} + \frac{12}{27} + \frac{27}{27} = 1\,{}^{13}/_{27}$

C2 Ex 1F

11 $g(x) = 2x^3 + 9x^2 - 6x - 5$

a) factorise $g(x)$ try to find p such that $g(p) = 0$

try $x=1$: $g(1) = 2(1)^3 + 9(1)^2 - 6(1) - 5 = 2+9-6-5 = 0$.
so $(x-1)$ is a factor. Now use division to reduce $g(x)$ to a quadratic that you can factorise.

$$
\begin{array}{r}
2x^2 + 11x + 5 \\
x-1 \,\big)\, 2x^3 + 9x^2 - 6x - 5 \\
-(2x^3 - 2x^2) \\
11x^2 - 6x \\
-(11x^2 - 11x) \\
5x - 5 \\
-(5x - 5) \\
0
\end{array}
$$

Now $2x^2 + 11x + 5 = (2x+1)(x+5)$

So $g(x) = (x-1)(2x+1)(x+5)$

b) Hence solve $g(x) = 0$ so $x = 1$, $x = -\frac{1}{2}$ or $x = -5$

C2 Ex1F

12 The remainder obtained when $f(x) = x^3 - 5x^2 + px + 6$ is divided by $(x+2)$ is equal to the remainder obtained when $f(x)$ ... is divided by $(x-3)$. Find the value of p.

$$f(-2) = f(3) \Rightarrow (-2)^3 - 5(-2)^2 + p(-2) + 6 = (3)^3 - 5(3)^2 + p(3) + 6$$

$$\Rightarrow -8 - 20 - 2p + 6 = 27 - 45 + 3p + 6$$

$$\Rightarrow -2p - 22 = -12 + 3p$$

$$\Rightarrow -10 = 5p$$

$$\Rightarrow p = -2.$$

C2 Ex1F

13 The remainder obtained when $f(x) = x^3 + dx^2 - 5x + 6$ is divided by $(x-1)$ is twice the remainder when ... $f(x)$ is divided by $(x+1)$. Find the value of d.

$$f(1) = 2f(-1) \Rightarrow (1)^3 + d(1)^2 - 5(1) + 6 = 2\left[(-1)^3 + d(-1)^2 - 5(-1) + 6\right]$$

$$\Rightarrow 1 + d - 5 + 6 = 2\left[-1 + d + 5 + 6\right]$$

$$\Rightarrow d + 2 = 2(d + 10)$$

$$\Rightarrow d + 2 = 2d + 20$$

$$\Rightarrow -18 = d$$

C2 Ex 1F

14a) Show that $(x-2)$ is a factor of $f(x) = x^3 + x^2 - 5x - 2$.

$(x-2)$ is a factor of $f(x) \Leftrightarrow f(2) = 0$ by the Factor Theorem.

say which rule you are using.

Find $f(2)$.

$$
\begin{aligned}
& f(2) = (2)^3 + (2)^2 - 5(2) - 2 \\
\Rightarrow\ & f(2) = 8 + 4 - 10 - 2 \\
\Rightarrow\ & f(2) = 12 - 12 \\
\Rightarrow\ & f(2) = 0 \\
\Rightarrow\ & (x-2) \text{ is a factor of } f(x). \ \square
\end{aligned}
$$

show plenty of clear steps

'$\square$' is a shorthand for "which is what I was trying to prove".

(b) on next page...

*"use part (a) or do it the hard way"

⊕ Hint: some of the solutions won't be whole numbers or fractions.

C2 Ex 1F

14 b) Hence, or otherwise*, find the exact⊕ solutions of the equation $f(x)=0$

We know that:

$f(x) = x^3 + x^2 - 5x - 2$ and $f(x) = (x-2)(Ax^2 + Bx + C)$ from (a)

$$
\begin{array}{r}
x^2 + 3x + 1 \\
x-2 \,\overline{)\, x^3 + x^2 - 5x - 2} \\
-(x^3 - 2x^2) \\
\hline
3x^2 - 5x \\
-(3x^2 - 6x) \\
\hline
x - 2 \\
-(x - 2) \\
\hline
0
\end{array}
$$

Now $x^2 + 3x + 1$ doesn't factorise as we predicted ⊕ above, so either complete the square on $x^2 + 3x + 1 = 0$

$\Rightarrow (x + 3/2)^2 - 9/4 + 4/4 = 0$

$\Rightarrow (x + 3/2)^2 - 5/4 = 0$

$\Rightarrow (x + 3/2)^2 = 5/4$

$\Rightarrow x + 3/2 = \pm\sqrt{5}/2$

$\Rightarrow x = \dfrac{-3 \pm \sqrt{5}}{2}$ exact form

Just quickly check for yourself that this surd won't simplify...

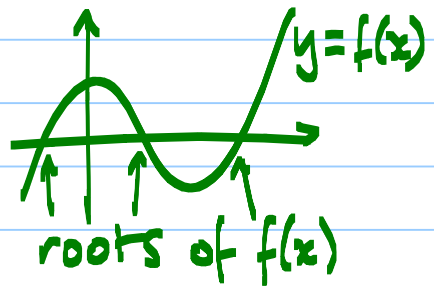
So our solutions are $x = 2$, $x = \dfrac{-3+\sqrt{5}}{2}$ or $x = \dfrac{-3-\sqrt{5}}{2}$

don't forget the easy one.

C2 Ex 1F

15 Given that -1 is a root* of the equation† $2x^3 - 5x^2 - 4x + 3$, find the two positive roots.

* A root of an equation $f(x) = 0$ is also called a 'solution'. The same value is sometimes called a zero of a function. Whatever you call it, it's where the curve of $y = f(x)$ crosses the x-axis.



† Excuse me for being pedantic, but $2x^3 - 5x^2 - 4x + 3$ is a function not an equation. So if we're being fussy this Exam Question (!) should have either said:

"Given that -1 is a zero of the function $2x^3 - 5x^2 - 4x + 3$, find..."

or "Given that -1 is a root of the equation $2x^3 - 5x^2 - 4x + 3 = 0$, find..."

C2 Ex 1F

15 Given that -1 is a root of the equation (!) $2x^3-5x^2-4x+3$, find the two positive roots.

-1 is a root of $f(x)=0 \Leftrightarrow f(-1)=0 \Leftrightarrow (x+1)$ is a factor of $f(x)$ by the factor theorem.

$$
\begin{array}{r|l}
 & 2x^2 \quad -7x \quad +3 \\
\hline
x+1 & 2x^3 - 5x^2 - 4x + 3 \\
 & -(2x^3 + 2x^2) \\
\hline
 & \quad -7x^2 - 4x \\
 & \quad -(-7x^2 - 7x) \\
\hline
 & \qquad 3x + 3 \\
 & \qquad -(3x+3) \\
\hline
 & \qquad\quad 0
\end{array}
$$

$2x^2 - 7x + 3 = (2x-1)(x-3)$

so the other roots are

$x = 1/2$ and $x = 3$